Lecture: 3-5 Implicit Differentiation (Part 2)

Example 1: Review. Find $\frac{dy}{dx}$ by implicit differentiation.

(a) $x^2 - xy - y^2 = 1$

$$2x - 1y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$

$$2x - y = x\frac{dy}{dx} + 2y\frac{dy}{dx}$$

$$2x - y = \frac{dy}{dx}(x + 2y)$$

$$\boxed{\frac{dy}{dx} = \frac{2x - y}{x + 2y}}$$

(b) $\sin(x+y) = 2x - 2y$

$$\cos(x+y)\cdot\left(1 + \frac{dy}{dx}\right) = 2 - 2\frac{dy}{dx}$$

$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = 2 - 2\frac{dy}{dx}$$

$$\cos(x+y)\frac{dy}{dx} + 2\frac{dy}{dx} = 2 - \cos(x+y)$$

$$\frac{dy}{dx}\left(\cos(x+y) + 2\right) = 2 - \cos(x+y)$$

$$\boxed{\frac{dy}{dx} = \frac{2 - \cos(x+y)}{\cos(x+y) + 2}}$$

Example 2 Find all points on the curve $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

determine where $dy/dx = 1$!

$$2x + 4y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{4y} = \frac{-x}{2y}$$

So $-\frac{x}{2y} = 1$ or $x = -2y$ also $x^2 + 2y^2 = 1$ (original equation)

And $(-2y)^2 + 2y^2 = 1$

$$6y^2 = 1$$

$$y^2 = 1/6$$

$$y = \pm 1/\sqrt{6}$$

using $x = -2y$

if $y = 1/\sqrt{6}$, $x = -2/\sqrt{6}$

if $y = -1/\sqrt{6}$, $x = 2/\sqrt{6}$

$\boxed{\text{points: } (-2/\sqrt{6}, 1/\sqrt{6}) \text{ and } (2/\sqrt{6}, -1/\sqrt{6})}$

Example 3: If $g(x) + x \sin g(x) = 3x^2 + 1$ and $g(1) = 0$ find $g'(1)$.

$$g'(x) + 1 \sin(g(x)) + x \cos(g(x))\, g'(x) = 6x$$
$$g'(x) + x \cos(g(x))\, g'(x) = 6x - \sin(g(x))$$
$$g'(x)(1 + x \cos(g(x))) = 6x - \sin(g(x))$$
$$g'(x) = \frac{6x - \sin(g(x))}{1 + x\cos(g(x))}$$
$$g'(1) = \frac{6 - \sin(g(1))}{1 + 1\cos(g(1))} = \frac{6 - \sin 0}{1 + \cos 0} = \frac{6-0}{1+1} = \boxed{3}$$

Derivatives of Inverse Trigonometric Functions

Implicit differentiation is also used to derive formulas for derivatives of inverse functions.

Example 4: Find the derivatives of the following functions.

(a) $y = \sin^{-1} x$

$$\sin y = \sin(\sin^{-1} x)$$
$$\sin y = x$$
$$\cos y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = 1/\cos y \;;$$
now $\cos^2 y + \sin^2 y = 1$

and $\cos^2 y + x^2 = 1$

$$\cos^2 y = 1 - x^2$$
$$\cos y = \sqrt{1-x^2}$$
$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}}$$

(b) $y = \tan^{-1} x$

$$\tan y = \tan(\tan^{-1} x)$$
$$\tan y = x$$
$$\sec^2 y \cdot \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} \;;$$
recall $1 + \tan^2 y = \sec^2 y$

so $1 + x^2 = \sec^2 y$

$$\boxed{\frac{dy}{dx} = \frac{1}{1+x^2}}$$

Example 5: Using implicit differentiation find the derivative of $y = \cos^{-1} x$.

$$\cos y = x$$
$$-\sin y \frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{-1}{\sin y}$$
$$\left[\begin{array}{l} \cos^2 y + \sin^2 y = 1 \\ \sin y = \sqrt{1 - \cos^2 y} \end{array}\right]$$
$$\boxed{\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}}$$

Derivatives of Inverse Trigonometric Functions:

- $\frac{d}{dx}(\sin^{-1} x) = \underline{1/\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \underline{-1/\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \underline{1/(1+x^2)}$

Example 6: Differentiate the following functions.

(a) $y = \cos^{-1}(3x + 5)$

$$y' = \frac{-1}{\sqrt{1-(3x+5)^2}} \cdot \frac{d}{dx}(3x+5)$$

$$= \frac{-3}{\sqrt{1-(9x^2+30x+25)}}$$

$$= \boxed{-3/\sqrt{-9x^2-30x-24}}$$

(b) $y = \arctan 2x$

$$y' = \frac{1}{1+(2x)^2} \cdot \frac{d}{dx} 2x$$

$$= \boxed{\frac{2}{1+4x^2}}$$

Example 7: Differentiate the following functions.

(a) $f(x) = \arcsin(\sqrt{x})$

$$f'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{d}{dx} x^{1/2}$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2} x^{-1/2}$$

$$= \boxed{\frac{1}{2\sqrt{x}\sqrt{1-x}}}$$

(b) $g(x) = \tan^{-1}(x - \sqrt{1 + x^2})$

$$g'(x) = \frac{1}{1+(x-\sqrt{1+x^2})^2} \cdot \frac{d}{dx}(x-\sqrt{1+x^2})$$

$$= \frac{1}{1+x^2-2x\sqrt{1+x^2}+(1+x^2)} \cdot \left(1-\frac{1}{2}(1+x^2)^{-1/2} \cdot 2x\right)$$

$$= \frac{1}{2+2x^2-2x\sqrt{1+x^2}}\left(1-\frac{x}{\sqrt{1+x^2}}\right)$$

$$= \frac{1}{2+2x^2-2x\sqrt{1+x^2}}\left(\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}}\right)$$

$$= \frac{\sqrt{1+x^2}-x}{(2+2x^2)\sqrt{1+x^2}-2x(1+x^2)}$$

take out the greatest common factor of $2+2x^2$.

$$= \frac{\sqrt{1+x^2}-x}{2(1+x^2)(\sqrt{1+x^2}-x)} = \boxed{\frac{1}{2(1+x^2)}}$$

Example 8: Differentiate the following functions.

(a) $y = x^2 \tan^{-1}\sqrt{x}$

$$y' = 2x\tan^{-1}(\sqrt{x}) + x^2 \cdot \frac{1}{1+\sqrt{x}^2} \cdot \frac{1}{2}x^{-1/2}$$

$$= \boxed{2x\tan^{-1}(\sqrt{x}) + \frac{x^2}{2\sqrt{x}(1+x)}}$$

(b) $y = x\sin^{-1}x + \sqrt{1-x^2}$

$$y' = 1\sin^{-1}x + x\cdot\frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$$

$$= \boxed{\sin^{-1}x}$$

Example 9: The *van der Waals* equation for n moles of a gas is

$$\left(P + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \Rightarrow PV - nbP + \frac{n^2a}{V} - \frac{n^3ab}{V^2} = nRT$$

where P is the pressure, V is the volume, and T is the temperature of the gas. The constant R is the universal gas constant and a and b are constants that are characteristic of a particular gas.

(a) If T remains constant, use implicit differentiation to find dV/dP. (V is like y, P is like x)

$$1\cdot V + P\frac{dV}{dP} - nb + n^2a(-1)V^{-2}\cdot\frac{dV}{dP} - n^3ab(-2)V^{-3}\frac{dV}{dP} = 0$$

$$V + P\frac{dV}{dP} - nb - \frac{n^2a}{V^2}\frac{dV}{dP} + \frac{2n^3ab}{V^3}\frac{dV}{dP} = 0$$

$$V^3\left(P\frac{dV}{dP} - \frac{n^2a}{V^2}\frac{dV}{dP} + \frac{2n^3ab}{V^3}\frac{dV}{dP}\right) = (nb - V)V^3$$

$$(V^3P - Vn^2a + 2n^3ab)\frac{dV}{dP} = nbV^3 - V^4$$

$$\boxed{\frac{dV}{dP} = \frac{nbV^3 - V^4}{V^3P - Vn^2a + 2n^3ab}}$$

(b) Find the rate of change of volume with respect to pressure of 1 mole of carbon dioxide at a volume of $V = 10$ L and a pressure of $P = 2.5$ atm. Use $a = 3.592$ L^2-atm/mole^2 and $b = 0.04267$ L/mole. $n=1$

$$\frac{dV}{dP} = \frac{(1(0.04267)10^3 - 10^4)}{(10^3(2.5) - 10\cdot 1^2(3.592) + 2\cdot 1^3(3.592)(0.04267))}$$

$$\approx -4.0405 \ \text{L/units of pressure}$$